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PHASE SPACE DILUTION IN THREE-TURN INJECTION

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The injection scheme proposed for the 100 GeV storage ring by Keil et al[1,2] envisages injection in three turns, using horizontal phase space. It is assumed that a set of kicker magnets displaces the equilibrium orbit at the septum during the injection process. The injected beam occupies an elliptical region of phase space, matched to the orbit characteristics of the beam, and tangent to the septum on the outside of the aperture.

If the location of the equilibrium orbit is at an appropriate position (see Fig. 1), the beam after one turn occupies an ellipse just tangent to the septum on the inside of the septum, and after two turns it is again just tangent on the inside of the septum. In (1) and (2) this is accomplished by making b, the distance of the equilibrium orbit from the septum (on the inside), just $\frac{1}{3}$ of the half-width of the beam, and the wave number ν of the ring is an integer $\pm\frac{1}{3}$.

After three turns of injection the kickers are turned off, returning the equilibrium orbit to its normal position. If this is more than (2a + s + b) (s = septum thickness) inside of the septum the whole beam now circulates on the inside of the aperture. The smallest matched phase space ellipse containing the beam has a half-width of 2a + s + b; therefore the average phase space density is reduced by a factor

$$F = \frac{3\ a^2}{(2a + s + b)^2} \qquad (1)$$

For $s = 0$, and with $b = \frac{1}{3}a$, this equals $27/49 = .55102$.

In this scheme it is necessary that the matched phase-space ellipse be "upright", i.e. that its axis be in the x and x' directions, which means that the ellipse parameter α must be equal to zero.

This implies that the injection point should be at the center of a symmetric straight section. However, in the 100 GeV NAL storage ring, the geometry dictates injection well downstream of such a point, i.e. at a point where α is negative. The matched phase space ellipse is then skew (Fig. 2), and the optimum injection conditions are different.

For the minimum size of the large ellipse surrounding three turns, the conditions are again that the injected beam be tangent to the septum on the outside, and both the first and second turns be tangent to it on the inside. We thus have the conditions:

Injected beam centered at $x = +a = \sqrt{\beta E/\pi}$ where E = emittance, β = amplitude function at injection point.

Equilibrium orbit at $x = -b$, value of b to be determined.

First turn centered at

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} -b \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu \end{pmatrix} \begin{pmatrix} a+b \\ 0 \end{pmatrix}$$

Tangency condition: $x_1 = -(a+s)$ (s = septum thickness) (2)

Second turn centered at

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} -b \\ 0 \end{pmatrix} + \begin{pmatrix} \cos 2\mu+\alpha\sin 2\mu & \beta\sin 2\mu \\ -\gamma\sin 2\mu & \cos 2\mu-\alpha\sin 2\mu \end{pmatrix} \begin{pmatrix} a+b \\ 0 \end{pmatrix}$$

Tangency condition

$$x_2 = -(a+s) \qquad (3)$$

There μ, the phase advance per revolution, must be chosen so that

$$\cos\mu+\alpha\sin\mu = \cos 2\mu+\alpha\sin 2\mu \qquad (4)$$

Solving (4) for α as a function of μ, we obtain

$$\alpha = \frac{\cos\mu-\cos 2\mu}{\sin 2\mu-\sin\mu} \; ; \; \cos\mu+\alpha\sin\mu = -\frac{1}{1-2\cos\mu} \qquad (5)$$

This gives

$$b = -a\frac{\cos\mu}{1-\cos\mu} + s\frac{1-2\cos\mu}{2(1-\cos\mu)} \qquad (6)$$

The smallest matched ellipse surrounding the three turns is centered on $(-b,0)$ and is tangent to the injection ellipse at $(a + d, 0)$, where $d = \frac{a}{\sqrt{1+\alpha^2}}$. Its width is $(\frac{a+b+d}{d})$ times that of the injection ellipse, and therefore the dilution factor is

$$F = \frac{3\,d^2}{(a+b+d)^2} = \frac{3\,a^2}{\left[(a+b)\sqrt{1+\alpha^2}+a\right]^2} .$$

With some algebra this becomes

$$F = 3\left(\frac{\sin^2\frac{\mu}{2}\cos\frac{\mu}{2}}{\sin^2\frac{\mu}{2}\cos\frac{\mu}{2} + \frac{s}{4a} + \frac{1}{2}}\right)^2 \qquad (7)$$

This quantity, as well as α from eq. (5), are plotted as functions of μ in Fig. 3. The graph covers only the range from 0 to 180°; the range from 180° to 360° (or -180° to 0°) can be obtained by noting that F is symmetrical and α antisymmetrical about $\mu=0$ and about $\mu=180^{\circ}$. In Fig. 3, F is plotted for zero septum thickness as well as for septum thickness as well as for septum thickness $s = \frac{1}{4}a$ and $s = a$. The following should be noted:

For a given value of α (with sign given), there are (between -180° and 180°) three solutions (for example, with $\alpha = -1$, these are at -150°, -30°, and 90°, for $\alpha = +0.5$ they are at -102°, 18^{0}, 138^{0}). The efficiency F is different at these three points; the best efficiency for $\alpha = -1$ is obtained at 90° (0.515), at -150° $F = 0.318$, and at -30° $F = 0.039$.

The maximum efficiency is obtained, not at $\alpha = 0$, but at $\alpha = \pm 0.283$.

The efficiency is not a very sensitive function of septum thickness. The maximum efficiency is $F = 0.567$ for $s = 0$, 0.495 for $s/a = .25$, 0.345 for $s/a = 1$, and 0.232 for $s/a = 2$ (when the septum is as wide as the beam).

A similar analysis can be obtained for four-turn injection. When this is done, the results are:

$$\alpha = \tan 2\mu$$

$$F = \frac{4(\cos\mu-\cos2\mu)}{(2+s/a+\cos\mu-\cos2\mu)^2}$$

(valid for $-120^{\circ} < \mu < 120^{\circ}$)

Maximum F obtained for $\cos\mu = 0.25$ ($\mu = \pm 75.5$ degrees; $\alpha = \pm .553$); $F_{max} = .5184$ with $s = 0$

For $\alpha = 0$, $\mu = 90^{\circ}$, $F\ (s = 0) = 4/9$.

References

1 Keil, Montague, Schnell and Sessler, FN-168 (8/27/68).

2 E. Keil, FN-169 (8/27/68).

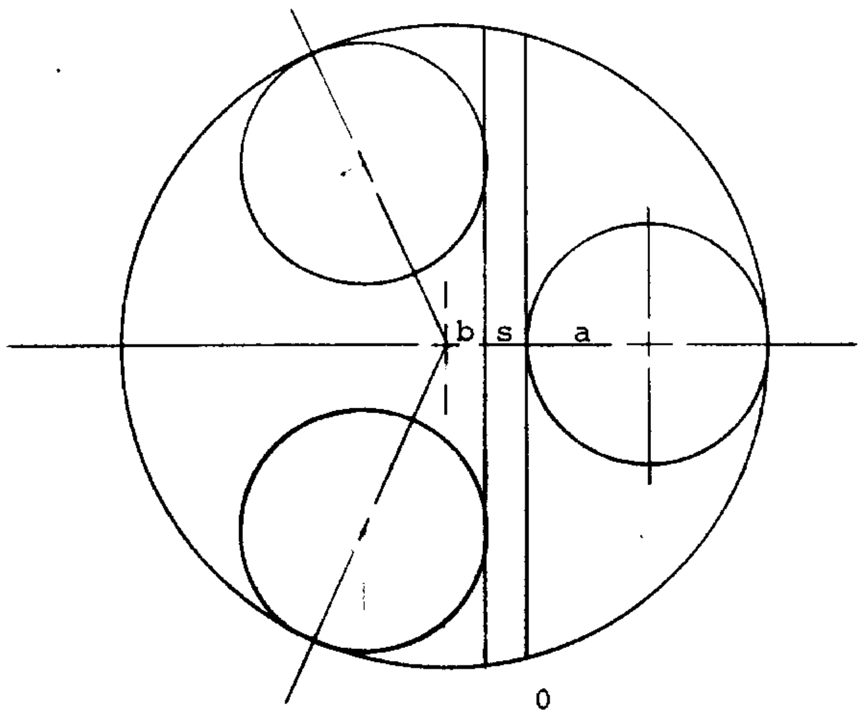

Fig. 1. Phase Space of three injected turns, phase ellipse circular

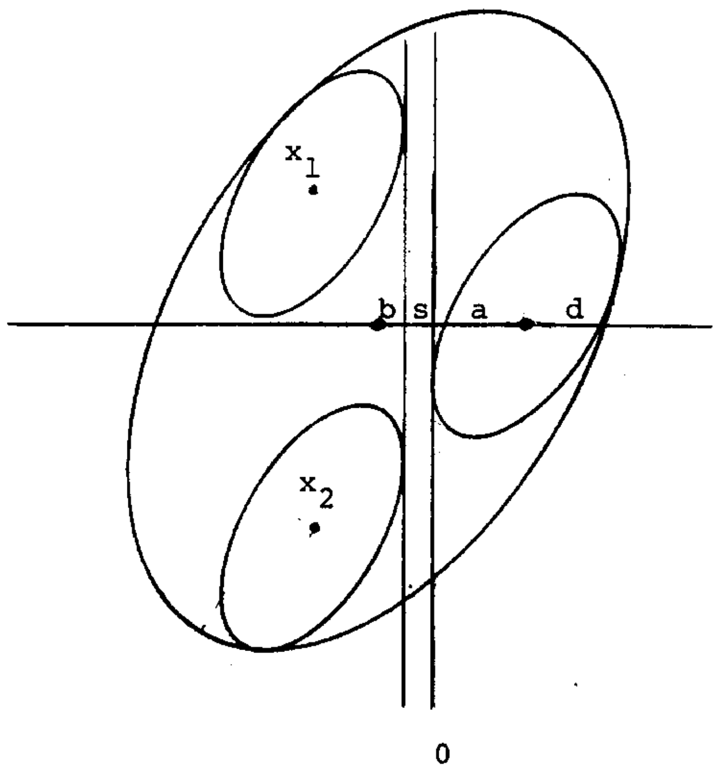

Fig. 2. Phase Space with skewed ellipse

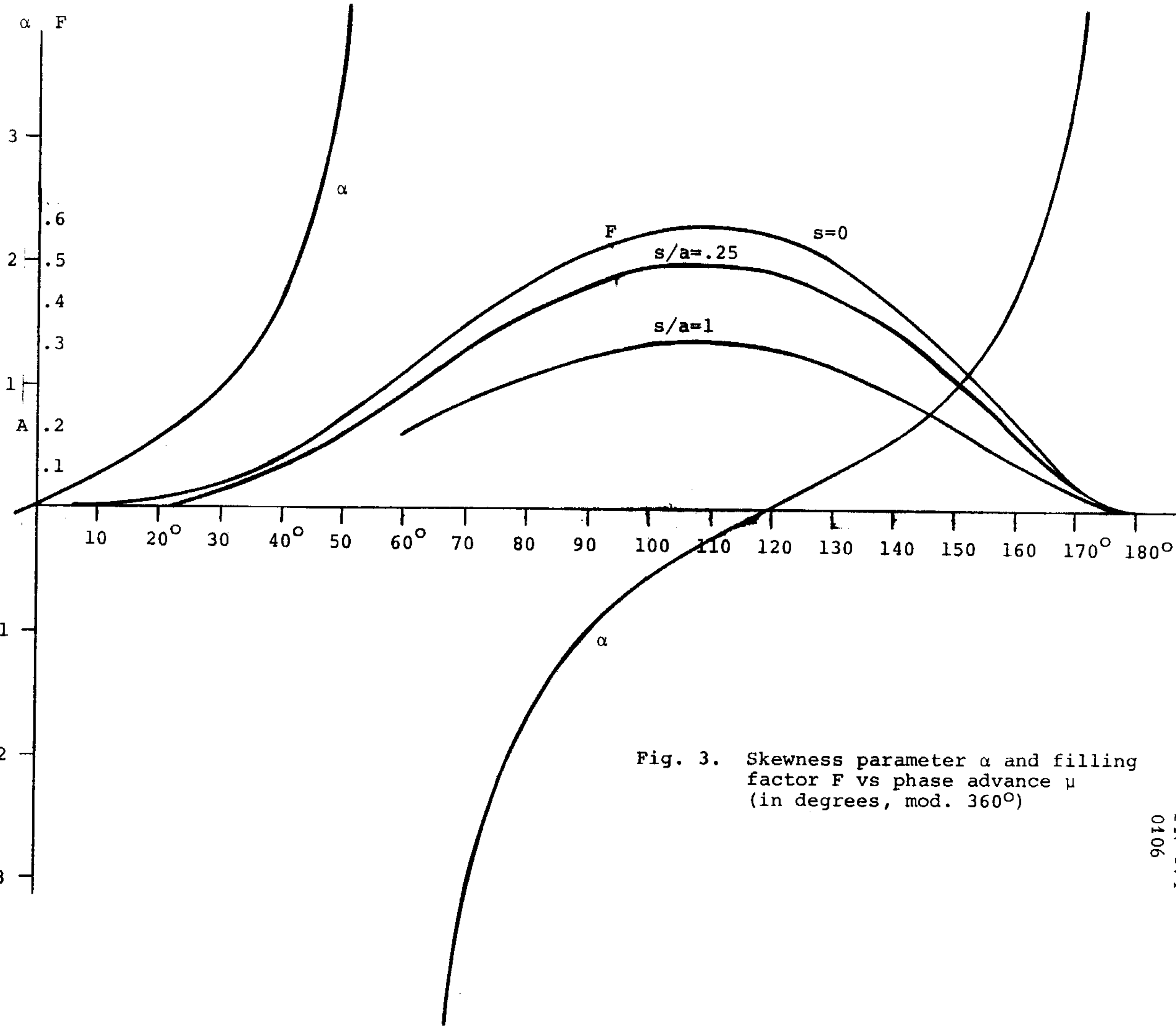

Fig. 3. Skewness parameter α and filling factor F vs phase advance μ (in degrees, mod. 360°)